MATHEMATICS OF INFINITY

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Посвящается памяти

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Intuitive considerations

Through the technique of so called lazy evaluation, it has become possible to compute with infinite objects of various kinds. The most typical and wellknown example is that of a stream

$$(a_0,(a_1,(a_2,\dots))),$$

which at any finite stage of its development looks like an initial segment of an ordinary list

$$(a_0,(a_1,(a_2,\dots(a_{n-1},\mathrm{nil})\dots))),$$

but differs by proceeding indefinitely. Another example is obtained by conceiving of an infinite binary sequence as an infinite composition of two unary constructors

$$0(1(0(\dots))),$$

rather than, as is customary, as a function from the set of natural numbers to the two element set. An even simpler example, the simplest possible, in a way, which will play a central role in the following, is the infinite natural number

$$s(s(s(\dots))),$$

the successor of the successor of the successor of etc. in infinitum. No other mathematical object, if only we can understand it as such, deserves better to be denoted by the traditional infinity symbol ∞.[1]

[1] Introduced by J. Wallis, De Sectionibus Conicis, Nova Methodo Expositis, Tractatus, Oxford, 1655, in the laconic parenthesis (esto enim ∞ nota numeri infiniti;), apparently without worrying about its meaningfulness.

All the preceding examples are examples of infinite elements of sets. But we may also let sets be infinite, not in the usual sense of containing infinitely many elements, but in the sense of having infinite depth, or proceeding indefinitely, like

$$A \times (A \times (A \times \ldots)),$$

where the set A may itself be infinitely proceeding, or

$$((\ldots) + (\ldots)) + ((\ldots) + (\ldots)).$$

The latter set is the disjoint union of two sets, each of which is the disjoint union of two sets, each of which is the disjoint union of two sets, etc. in infinitum. It looks very much like the Cantor set.

If we can conceive of infinitely proceeding sets, we can certainly also conceive of infinitely proceeding propositions: because of the correspondence between propositions and sets, there is no substantial difference. A typical example is

$$\sim\sim\sim\ldots = ((\ldots \supset \perp) \supset \perp) \supset \perp,$$

the negation of the negation of the negation of etc. in infinitum. Since such an infinitely proceeding proposition has no bottom that you reach in a finite number of steps, it is not at all immediately clear what it should mean for it to be true. Nor does it seem clear whether this particular one ought to come out true or false. (To anticipate matters, on the interpretation that I shall adopt, it will come out false.)

An infinite object of yet another kind is the iterative set, that is, set in the sense of the cumulative hierarchy,

$$\{\{\{\ldots\}\}\}.$$

It is the singleton set whose only element is the singleton set whose only element is the singleton set etc. in infinitum. No such set exists, of course, in the usual cumulative hierarchy, but it is as simple as possible an example of a nonwellfounded set in the sense of Aczel.[2] Finally, you may even conceive of nonmathematical examples of infinite objects, like the wellknown picture of the artist painting his own portrait.

When you start thinking about infinite objects, like the aforementioned ones, you soon realize that they are maybe not so novel creatures after all. We have also old examples, like infinite decimal fractions

$$a_0.a_1a_2a_3\ldots = a_0 + \frac{1}{10}(a_1 + \frac{1}{10}(a_2 + \frac{1}{10}(a_3 + \ldots)))$$

and infinite continued fraction expansions

$$a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ldots,}}}$$

which proceed indefinitely in just the same way as the streams of the computer scientist. And we all know the mathematics that has been developed in order to deal rigorously with these particular infinite objects, namely, calculus, or analysis, in its

[2] P. Aczel, Non-Well-Founded Sets, CSLI Lecture Notes, Number 14, Stanford University, 1988.

various forms. Originally, from its inception to the time of Euler, approximately, infinitesimal calculus was really a calculus of infinites and infinitesimals, that is, of infinitely large and infinitely small quantities, like

$$\infty,\ 2\infty,\ 3\infty,\ \infty^2,\ \infty^3,\ \infty^\infty,\ \infty^{\infty^\infty},\ \ldots$$

and

$$\frac{1}{\infty},\ \frac{1}{\infty^2},\ \frac{1}{\infty^3},\ \ldots,$$

respectively. But, when it was put on a rigorous basis by Cauchy and Weierstrass, the infinites and infinitesimals were gradually eliminated in favour of the notion of limit, earlier introduced by d'Alembert, although traces of them still remain in some of our notations, like

$$\lim_{n=\infty} a_n,\ \sum_{n=1}^{\infty} a_n,\ \prod_{n=1}^{\infty} a_n,\ \ldots$$

Usually, though, we take great pains to explain that ∞ makes no sense by itself, that is, is no detachable part of the notation for a limit, an infinite sum, product, or the like, and would have difficulties in interpreting an expression like

$$(1 + \frac{1}{\infty})^{\infty}.$$

Only during the period of the last thirty years has there been a resurgence of interest in infinite and infinitesimal numbers as a result of Abraham Robinson's conception of his nonstandard analysis.[3] There is also the slightly earlier but less wellknown infinitesimal calculus of Schmieden and Laugwitz, which succeeds in making sense of expressions containing the infinity symbol in a much more elementary and constructive way.[4] Otherwise, all these various forms of analysis are classical theories. On the intuitionistic side, we have, on the one hand, the straightforward constructivization of analysis in the style of Cauchy and Weierstrass carried out by Bishop,[5] and, on the other hand, Brouwer's much more radically novel idea of choice sequences.[6] It is one of the purposes of the present work to show that the introduction of choice sequences is an intuitionistic version of the formation of reduced products in nonstandard analysis. (Observe that the theory of choice sequences antedates nonstandard analysis by forty years.)

The two recent theories that have been contrived precisely for dealing with streams and other infinite objects, like the ones mentioned in the beginning, are Scott's theory of domains for denotational semantics and Aczel's theory of nonwellfounded

[3] A. Robinson, Non-standard Analysis, North-Holland Publishing Company, Amsterdam, 1966.

[4] C. Schmieden and D. Laugwitz, Eine Erweiterung der Infinitesimalrechnung, Mathematische Zeitschrift, Vol. 69, 1958, pp. 1-39. See also the book by D. Laugwitz, Infinitesimalkalkül, Eine elementare Einführung in die Nichtstandard-Analysis, Bibliographisches Institut, Mannheim, 1978, and the further references given there.

[5] E. Bishop, Foundations of Constructive Analysis, McGraw-Hill Book Company, New York, 1967.

[6] L. E. J. Brouwer, Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil: Allgemeine Mengenlehre, Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Sect. 1, Vol. 12, No. 5, 1918, pp. 3-43.

sets.[7] Of these, domain theory indeed succeeds in making sense of streams and similar infinite objects, but it does not succeed in interpreting logic in a nontrivial way which harmonizes with the interpretation of sets as domains, elements of a set as points of a domain, functions from one set to another as approximable mappings in Scott's sense, etc. Now, since all the set theoretical laws, as formalized in my type theory,[8] are validated in the domain interpretation, the interpretation of logic might seem as simple as it could possibly be: just stick to the interpretation of propositions as sets, truth as nonemptiness, etc. But what happens? We indeed get an interpretation of logic satisfying all the usual laws of intuitionistic logic: the only trouble with it is that it trivializes in the sense that it makes every proposition, even absurdity, come out true. The reason is that every domain contains an element, namely, the bottom element. Hence, if propositions, like sets, are interpreted as domains and truth as nonemptiness, every proposition comes out true.

The principal aim of domain theory is to make proper mathematical sense of the fixed point operator

$$\mathrm{fix}(f) = f(f(f(\ldots))).$$

We have here yet another example of an infinite object to add to the long list in the beginning. Now, written in type theoret-

[7] D. Scott, Domains for denotational semantics, Lecture Notes in Computer Science, Vol. 140, Automata, Languages and Programming, Edited by M. Nielsen and E. M. Schmidt, Springer-Verlag, Berlin, 1982, pp. 577-613, and P. Aczel, op. cit.

[8] P. Martin-Löf, Intuitionistic Type Theory, Bibliopolis, Napoli, 1984.

ical notation, the formal laws for the fixed point operator that domain theory seeks to satisfy, and succeeds in satisfying, are

$$\frac{\begin{array}{c}(x \in A)\\ f(x) \in A\end{array}}{\mathrm{fix}(f) \in A}, \qquad \frac{\begin{array}{c}(x \in A)\\ f(x) \in A\end{array}}{\mathrm{fix}(f) = f(\mathrm{fix}(f)) \in A}.$$

They say that every approximable mapping from a domain into itself has a fixed point. On the other hand, for an arbitrary set A, we can certainly derive

$$\begin{cases} \mathrm{id}(x) \in A \ (x \in A), \\ \mathrm{id}(x) = x \in A \ (x \in A) \end{cases}$$

by explicit definition in standard type theory. Hence, if the formal laws for the fixed point operator are adjoined to standard type theory, we can derive

$$\mathrm{fix}(\mathrm{id}) \in A$$

for an arbitrary set A, also to be thought of as a proposition. In particular,

$$\mathrm{fix}(\mathrm{id}) \in N_0 = \perp .$$

Thus type theory becomes inconsistent when the formal laws for the fixed point operator are adjoined to it.

We might try to avoid this inconsistency by only requiring a function from a nonempty set into itself to have a fixed point. The previous laws for the fixed point operator then get modified into

$$\frac{a \in A \qquad \begin{matrix}(x \in A)\\ f(x) \in A\end{matrix}}{\mathrm{fix}(a,f) \in A}, \qquad \frac{a \in A \qquad \begin{matrix}(x \in A)\\ f(x) \in A\end{matrix}}{\mathrm{fix}(a,f) = f(\mathrm{fix}(a,f)) \in A}.$$

These laws, in turn, are readily seen to have the same effect as introducing an infinite natural number, governed by the axioms

$$\begin{cases} \infty \in N, \\ \infty = s(\infty) \in N. \end{cases}$$

Indeed, given the modified fixed point operator, we can define infinity by putting

$$\infty = \mathrm{fix}(0,s) \in N,$$

where $s(x) \in N$ $(x \in N)$ is the successor function, and, conversely, we can define the modified fixed point operator by performing an ordinary recursion up to infinity,

$$\mathrm{fix}(a,f) = \mathrm{rec}(\infty,a,(x,y)f(y)) \in A.$$

That the second rule for the modified fixed point operator becomes satisfied follows from the axiom $\infty = s(\infty) \in N$ and the second recursion equation,

$$\begin{aligned} \mathrm{fix}(a,f) &= \mathrm{rec}(\infty,a,(x,y)f(y)) \\ &= \mathrm{rec}(s(\infty),a,(x,y)f(y)) \\ &= f(\mathrm{rec}(\infty,a,(x,y)f(y))) \\ &= f(\mathrm{fix}(a,f)) \in A. \end{aligned}$$

This seems fine, but, although we no longer get the fixed point of the identity function as an element of the empty set alias proof of falsehood, the system is still inconsistent, because

$$\sim I(N, x, s(x)) \text{ true } (x \in N)$$

is readily proved by mathematical induction in standard type theory, from which

$$\sim I(N, \infty, s(\infty)) \text{ true}$$

follows by instantiation once we allow $\infty \in N$. On the other hand,

$$I(N, \infty, s(\infty)) \text{ true}$$

follows of course from the definitional, or intensional, equality $\infty = s(\infty) \in N$. Thus we have reached a contradiction, which shows that the circular definition $\infty = s(\infty) \in N$ is inadmissible.

Aczel's approach in his theory of nonwellfounded sets is to avoid contradiction in introducing nonwellfounded sets, like

$$\Omega = \{\{\{\ldots\}\}\},$$

by relaxing the axioms of standard set theory, namely, by giving up the foundation axiom and replacing it by his antifoundation axiom, which is in contradiction with it. In the case of arithmetic rather than set theory, this would mean giving up the principle of mathematical induction. My own intuition has been that, on the contrary, all the laws for the standard, or wellfounded, objects should remain valid for the nonstandard, nonwellfounded, or infinitely proceeding, objects, which is to say that my thoughts have gone rather in the direction of nonstandard arithmetic and analysis. Also, the idea of getting the fixed point operator, and thereby all sorts of infinite objects, from the single infinite natural number

$$\infty = s(s(s(\ldots)))$$

is very reminiscent of the fundamental property of a nonstandard model of arithmetic that it contain a natural number which is greater than

$$0,\ s(0),\ s(s(0)),\ \ldots,$$

that is, which is greater than all standard natural numbers.

How are we then to make proper mathematical sense of the infinite? To get out of the dead end that we have reached, we must turn to the theory of choice sequences. A choice sequence

$$\alpha = f_0(f_1(f_2(\ldots)))$$

is determined by a noncircular but nonwellfounded definitional process

$$\begin{aligned} \alpha = \alpha_0 = f_0(\alpha_1),& \\ \alpha_1 = f_1(\alpha_2),& \\ \alpha_2 = f_2(\alpha_3),& \\ \ldots& \end{aligned}$$

Here f_i is a function from A_{i+1} to A_i, where A_i is a nonempty set, that is, a set containing an element a_i, for i = 0, 1, 2, etc. In particular, A_0 is the set to which the choice sequence $\alpha = \alpha_0$ belongs. The intuition is the following. At the zeroth stage, we know nothing about α_0 except that it belongs to the set A_0. Thus α_0 is simply a variable ranging over A_0. Then, when we ask what element of A_0 that α_0 is, we get to know that $\alpha_0 = f_0(\alpha_1)$, where α_1 belongs to A_1. At this first stage, α_1 is freely variable, whereas α_0 has become partially determined: it is neither freely variable nor constant but something midway in between. At the second stage, we ask what element of A_1 that α_1 is, and get to know that $\alpha_1 = f_1(\alpha_2)$, where α_2 is an element of A_2. The information about α thereby gets refined from $f_0(\alpha_1)$ to $f_0(f_1(\alpha_2))$. In this way, the definitional process continues without end. The reason why the nonwellfoundedness of the definition will not lead to any inconsistency is that, at any finite stage, we can break it off by putting $\alpha_i = a_i$, where a_i is the element which shows the set A_i to be nonempty. The choice sequence α then gets approximated by $f_0(f_1(\ldots f_{i-1}(a_i)\ldots))$, which is a standard element of A_0. This notion of choice sequence is essentially due to Troelstra.[9] The only difference is that he has been concerned with the case when the functions, from which the choice sequence is obtained by infinite composition, are continuous functions on the Baire space.

Now, think of

$$\infty = s(s(s(\ldots)))$$

as a choice sequence, that is, as defined by the nonwellfounded definitional process

[9] A. S. Troelstra, Choice Sequences, A Chapter of Intuitionistic Mathematics, Clarendon Press, Oxford, 1977. See particularly Appendix C, pp. 152-160, and the references to earlier works given there.

$$\begin{aligned}\infty = \infty_0 = s(\infty_1), \quad & \\ \infty_1 = s(\infty_2), & \\ \infty_2 = s(\infty_3), & \\ \ldots & \end{aligned}$$

We are then lead to extend standard type theory by adjoining the axioms

$$\begin{cases} \infty_i \in \mathrm{N}, \\ \infty_i = s(\infty_{i+1}) \in \mathrm{N} \end{cases}$$

for i = 0, 1, etc., and to define $\infty \in \mathrm{N}$ by the explicit definition $\infty = \infty_0 \in \mathrm{N}$. It was only the accidental fact that the particular choice sequence $\infty = s(s(s(\ldots)))$ proceeds in the same way all the time that seduced us into making the circular definition $\infty = s(\infty) \in \mathrm{N}$, which we have seen to be inconsistent.

Make the explicit definition

$$\mathrm{fix}_i(a,f) = \mathrm{rec}(\infty_i, a, (x,y)f(y)) \in A,$$

where $a \in A$ and $f(x) \in A$ $(x \in A)$. Then $\mathrm{fix}_i(a,f)$ obeys the rules

$$\frac{a \in A \qquad \begin{matrix}(x \in A) \\ f(x) \in A\end{matrix}}{\mathrm{fix}_i(a,f) \in A}, \qquad \frac{a \in A \qquad\qquad \begin{matrix}(x \in A) \\ f(x) \in A\end{matrix}}{\mathrm{fix}_i(a,f) = f(\mathrm{fix}_{i+1}(a,f)) \in A}.$$

Conversely, given $\mathrm{fix}_i(a,f)$ as governed by these two rules, we can define $\infty_i \in \mathrm{N}$ by putting

$$\infty_i = \mathrm{fix}_i(0,s) \in \mathrm{N}$$

and thereby satisfy the axioms for ∞_i, i = 0, 1, etc. Thus the rules for $fix_i(a,f)$ and the axioms for ∞_i are formally equivalent. It is natural to refer to $fix_i(a,f)$ as the indexed fixed point operator.

It is important that $\infty_i \in N$, $i = 0, 1$, etc., is an external sequence of nonstandard natural numbers satisfying $\infty_i = s(\infty_{i+1}) \in N$, $i = 0, 1$, etc., because the axioms for a corresponding internal sequence, that is,

$$\begin{cases} \infty(x) \in N \ (x \in N), \\ \infty(x) = s(\infty(s(x))) \in N \ (x \in N), \end{cases}$$

lead of course to a contradiction, since they entail

$$(\forall x \in N)(\infty(s(x)) < \infty(x)) \text{ true},$$

which is in contradiction with the principle of mathematical induction.

Observe that the presence of $\infty \in N$ allows us to get a closed notation for the limit of an internal sequence of mathematical objects of whatever kind. For instance, if $A(x)$ is a set for $x \in N$, then

$$\lim_{x=\infty} A(x) = A(\infty)$$

is a set. Likewise, and most importantly, if $a(x) \in A(x)$ for $x \in N$, then

$$\lim_{x=\infty} a(x) = a(\infty) \in A(\infty).$$

Thus the limit operation $\lim\limits_{x=\infty}$ is expressed simply as the literal substitution of the infinity symbol ∞ for the variable x which tends to infinity. Since substitution is defined so as to commute with every other operation, this means that the limit operation also automatically does so. For instance, if $a(x) \in Q$ and $b(x) \in Q$ for $x \in N$, where Q is the set of rational numbers, then

$$\lim_{x=\infty} (a(x) + b(x)) = a(\infty) + b(\infty)$$

$$= \lim_{x=\infty} a(x) + \lim_{x=\infty} b(x) \in Q$$

holds by definition. This example shows two virtues of the nonstandard approach to analysis, namely, that limits always exist and can be expressed simply by substituting the infinity symbol for the variable which tends to infinity.

Let the predecessor and cut off subtraction functions be defined as usual by the recursion equations

$$\begin{cases} pd(0) = 0 \in N, \\ pd(s(a)) = a \in N, \end{cases}$$

$$\begin{cases} a - 0 = a \in N, \\ a - s(b) = pd(a - b) \in N, \end{cases}$$

that is, in type theoretical notation,

$$pd(a) = rec(a,0,(x,y)x) \in N,$$

$$a - b = rec(b,a,(x,y)pd(y)) \in N.$$

Then, from the axiom

$$\infty_i = s(\infty_{i+1}) \in N,$$

we get inversely

$$\infty_{i+1} = pd(\infty_i) \in N.$$

Iterated use of the latter equation yields

$$\infty_i = pd(\infty_{i-1}) = \ldots = pd^i(\infty) = \infty - s^i(0) \in N$$

for i = 0, 1, etc. Thus it is enough to introduce the single infinite number $\infty \in N$ and require it to satisfy the definitional equalities

$$\infty - s^i(0) = pd^i(\infty) = s(pd^{i+1}(\infty)) = s(\infty - s^{i+1}(0)) \in N.$$

The latter equalities, in turn, are equivalent to

$$\infty = s^i(pd^i(\infty)) = s^i(\infty - s^i(0)) \in N$$

for i = 0, 1, etc. So, although we can manage with a single new constant, we still need infinitely many axioms to characterize it.

From the axioms $\infty_i = s(\infty_{i+1}) \in N$ for i = 0, 1, etc., it is readily proved that each ∞_i is an infinite natural number in the sense of nonstandard arithmetic, which is to say that it exceeds all standard natural numbers. Indeed, we have

$$0 \leqslant x \text{ true } (x \in N),$$

and hence, by instantiation,

$$0 \leqslant \infty_{i+j} \text{ true}$$

for arbitrary i and j = 0, 1, etc. But the successor function is monotonic, so that we can conclude

$$s^j(0) \leqslant s^j(\infty_{i+j}) \text{ true.}$$

On the other hand, from the axioms $\infty_i = s(\infty_{i+1}) \in N$, there follow the definitional equalities

$$\infty_i = s^j(\infty_{i+j}) \in N.$$

Therefore we have

$$s^j(0) \leqslant \infty_i \text{ true}$$

for all $j = 0, 1$, etc., which shows that each ∞_i is infinite in the sense of nonstandard arithmetic.

Nonstandard extension of type theory.

Let there be given, in the system T of standard type theory, a projective system of nonempty sets

$$A_0 \xleftarrow[f_0]{} A_1 \xleftarrow[f_1]{} \cdots \xleftarrow[f_{i-1}]{} A_i \xleftarrow[f_i]{} A_{i+1} \xleftarrow[f_{i+1}]{} \cdots,$$

or, in the formal notation,

$$\begin{cases} A_i \text{ set}, \\ f_i(x_{i+1}) \in A_i \ (x_{i+1} \in A_{i+1}), \\ a_i \in A_i, \end{cases}$$

where $i = 0, 1$, etc. The case that we shall be especially interested in is when

$$\begin{cases} A_i = N, \\ f_i(x_{i+1}) = s(x_{i+1}) \in N \ (x_{i+1} \in N), \\ a_i = 0 \in N, \end{cases}$$

corresponding to the picture

$$N \xleftarrow[s]{} N \xleftarrow[s]{} \cdots \xleftarrow[s]{} N \xleftarrow[s]{} N \xleftarrow[s]{} \cdots$$

In the most general case, each A_i need not be just a single set but a whole context

$$x_{i1} \in A_{i1}, \ \ldots, \ x_{in_i} \in A_{in_i}(x_{i1},\ldots,x_{in_i-1}),$$

which is to say that

$$\begin{aligned}
&A_{i1} \text{ set},\\
&\quad\vdots\\
&A_{in_i}(x_{i1},\ldots,x_{in_i-1}) \text{ set } (x_{i1} \in A_{i1}, \ \ldots,\\
&\qquad\qquad x_{in_i-1} \in A_{in_i-1}(x_{i1},\ldots,x_{in_i-2})).
\end{aligned}$$

Then

$$x_i = x_{i1},\ldots,x_{in_i}$$

is to be interpreted as a sequence of variables, namely, as the sequence of variables occurring in that context,

$$f_i(x_{i+1}) = f_{i1}(x_{i+11},\ldots,x_{i+1n_{i+1}}),\ldots,f_{in_i}(x_{i+11},\ldots,x_{i+1n_{i+1}})$$

as a sequence of functions of several variables mapping the context at stage i+1 into the context at stage i, which means that

$$\begin{aligned}
&f_{i1}(x_{i+11},\ldots,x_{i+1n_{i+1}}) \in A_{i1},\\
&\qquad\quad\vdots\\
&f_{in_i}(x_{i+11},\ldots,x_{i+1n_{i+1}}) \in A_{in_i}(f_{i1}(x_{i+11},\ldots,x_{i+1n_{i+1}}),\ldots,\\
&\qquad\qquad\qquad\qquad f_{in_i-1}(x_{i+11},\ldots,x_{i+1n_{i+1}})),
\end{aligned}$$

in all cases for $x_{i+11} \in A_{i+11}, \ \ldots, \ x_{i+1n_{i+1}} \in A_{i+1n_{i+1}}(x_{i+11}, \ldots,x_{i+1n_{i+1}-1})$, and

$$a_i = a_{i1}, \ldots, a_{in_i}$$

as a sequence of elements

$$\begin{aligned} &a_{i1} \in A_{i1}, \\ &\quad\vdots \\ &a_{in_i} \in A_{in_i}(a_{i1}, \ldots, a_{in_i-1}), \end{aligned}$$

that is, as an instance of the context at stage $i = 0, 1$, etc. With this amount of vector notation, the general case can be reduced notationally to the special case that I started by considering, and shall continue to consider in the following. It is natural to speak with Troelstra of the general case as that of a network of interdependent choice sequences instead of just a single one.[10]

Now, extend the system T of standard type theory by adjoining the axioms for a single choice sequence $\alpha = f_0(f_1(f_2(\ldots)))$, that is,

$$\begin{cases} \alpha_i \in A_i, \\ \alpha_i = f_i(\alpha_{i+1}) \in A_i \end{cases}$$

for $i = 0, 1$, etc., and call the extension T_α. The new axioms may be interpreted as saying that the projective limit of the given projective system of nonempty sets is itself nonempty. This is a counterpart of the so called countable saturation principle of nonstandard analysis, which makes it natural to refer

[10] A. S. Troelstra, op. cit., p. 154.

to T_α not only as the nonstandard extension but also as the saturation of T.[11]

How is provability in the nonstandard extension T_α related to provability in the standard theory T? This question is answered by the following lemma.

Lemma (proof theoretic). A judgement is provable in T_α if and only if, at some stage $j = 0, 1$, etc., the judgement which is obtained from it by replacing each occurrence of α_i by $f_i(f_{i+1}(\ldots f_{j-1}(x_j)\ldots))$ is provable in T from the assumption $x_j \in A_j$.

The stage j up to which you need to go must of course be at least as great as the maximum of the indices of the constants α_i that occur in the judgement in question. It may even have to be strictly greater.

To prove the necessity of the condition, observe first that, because of the finiteness of a proof, a judgement is provable in T_α if and only if it is provable in T_j at some stage $j = 0, 1$, etc., where T_j is the finite extension of T obtained by adjoining the $2j+1$ axioms

$$\alpha_j \in A_j,$$

$$\begin{cases} \alpha_{j-1} \in A_{j-1}, \\ \alpha_{j-1} = f_{j-1}(\alpha_j) \in A_{j-1}, \end{cases}$$

$$\vdots$$

[11] S. Albeverio, J. E. Fenstad, R. Hoegh-Krohn, and T. Lindstrøm, Nonstandard Methods in Stochastic Analysis and Mathematical Physics, Academic Press, New York, 1986, p. 46.

$$\begin{cases} \alpha_0 \in A_0, \\ \alpha_0 = f_0(\alpha_1) \in A_0. \end{cases}$$

But for the crucial first of these axioms, this is nothing but a sequence of explicit definitions, successively defining α_{j-1}, α_{j-2}, etc., ultimately α_0 in terms of α_j. Indeed, we have

$$\alpha_i = f_i(f_{i+1}(\dots f_{j-1}(\alpha_j)\dots)) \in A_j$$

for $0 \leqslant i < j$. Thus T_j is a definitional extension of the theory which is obtained from T by adjoining the single axiom

$$\alpha_j \in A_j.$$

This being the only axiom which governs the constant α_j, we may as well replace it by a variable x_j, thereby transforming the axiom $\alpha_j \in A_j$ into the assumption $x_j \in A_j$. This proves the necessity of the condition.

The sufficiency of the condition is clear, because T_j is an extension of T, and, in T_j, we have access to the axiom

$$\alpha_j \in A_j$$

as well as the definitional equalities

$$\alpha_i = f_i(f_{i+1}(\dots f_{j-1}(\alpha_j)\dots)) \in A_i$$

for $0 \leqslant i < j$, in addition to the axioms of T, so that we can first substitute α_j for x_j and then replace $f_i(f_{i+1}(\dots f_{j-1}(\alpha_j)\dots))$ by α_i. Moreover, since T_α is an extension of T_j, provability in T_j entails provability in T_α.

Observe that the only rules of T, and hence of T_j and T_α, that we have used in the course of the proof are the substitution and equality rules. It is thus immaterial exactly what the proper axioms of T are.

By means of the lemma, the following proof theoretic version of the transfer principle is easily established.

Transfer principle (proof theoretic). Let A be a proposition expressed in the language of T. Then A can be proved to be true in T_α if and only if A can be proved to be true in T already.

The sufficiency of the condition is clear since T_α is an extension of T. To prove the necessity, assume that

$$A \text{ true}$$

is provable in T_α, that is, that

$$a \in A$$

is provable in T_α for some a. The proof expression a is of course in general nonstandard, but, by the lemma,

$$a = b(\alpha_j) \in A,$$

where

$$b(x_j) \in A \ (x_j \in A_j)$$

is provable in T already. Also, by assumption,

$$a_j \in A_j$$

is provable in T. Hence, by substitution, so is

$$b(a_j) \in A.$$

Suppressing the proof expression, we can conclude that

$$A \text{ true}$$

is provable in T already.

The transfer principle is a consequence of the possibility of approximating a nonstandard proof by a standard one. Applying it to the proposition $\perp = N_0$, which is certainly expressed in the language of T, we can conclude that T_α is consistent relative to T. This relative consistency proof is of a very elementary nature. But, to conclude that T_α is consistent outright, we need to combine it with the semantic consistency proof for the standard theory T.[12]

Inductive limit interpretation

The preceding proof theoretic treatment of the nonstandard extension has an exact model theoretic counterpart. Let M be the standard model of the formal system T of standard type theory. When specifying the model M, as compared with when you specify the theory T, there is no difference in the symbols that you put down: the difference is only one of attitude, or point of view. When specifying M, every expression is to have its usual meaning, or intended interpretation, whereas, when specifying T, it is to be interpreted purely formalistically, that is, as standing for itself and not for its meaning. For example, among the objects in the standard model, there is the set N of natural numbers and the particular natural number 0, whereas, in the theory, there

[12] P. Martin-Löf, op. cit., pp. 69-70.

is the set expression N and the numerical expression 0. The difference in attitude is brought about by speaking of objects of the various semantical categories in the one case, and expressions of the corresponding syntactical categories in the other case.

The nonstandard model is built over the same projective system of nonempty sets, or, more generally, contexts, as the nonstandard theory, namely,

$$A_0 \xleftarrow[f_0]{} A_1 \xleftarrow[f_1]{} \dots \xleftarrow[f_{i-1}]{} A_i \xleftarrow[f_i]{} A_{i+1} \xleftarrow[f_{i+1}]{} \dots$$

The only difference is that, this time, the sequence of sets, mappings between them, and elements contained in them,

$$\begin{cases} A_i \text{ set}, \\ f_i(x_{i+1}) \in A_i \ (x_{i+1} \in A_{i+1}), \\ a_i \in A_i \end{cases}$$

for i = 0, 1, etc., are thought of as objects rather than expressions in the language of T.

Using category theoretic terminology and notation, let M_i be the comma model M over A_i, in symbols,

$$M_i = M/A_i.$$

A type in the sense of M_i is a family of types over A_i, that is, a type which has been made dependent on a variable which varies over A_i, and an object of the type in the sense of M_i is a function defined on A_i whose value for a certain argument is an object of the type in the family corresponding to that argument.

For example, a set in the sense of M_i is a family of sets over A_i, say

$$A(x_i) \text{ set } (x_i \in A_i),$$

and an element of that set in the sense of M_i is a function

$$a(x_i) \in A(x_i) \ (x_i \in A_i).$$

M_i is clearly a nonstandard model of T, and it becomes a model of T_i by interpreting the constant

$$\alpha_i \in A_i$$

simply as

$$x_i \in A_i \ (x_i \in A_i),$$

that is, as the identity function on A_i.

When passing from A_i to M/A_i, the given projective system of sets is transformed into the injective system of models

$$M/A_0 \xrightarrow[f_0^*]{} M/A_1 \xrightarrow[f_1^*]{} \cdots \xrightarrow[f_{i-1}^*]{} M/A_i \xrightarrow[f_i^*]{} M/A_{i+1} \xrightarrow[f_{i+1}^*]{} \cdots$$

Here f_i^* denotes composition with f_i. Thus f_i^* takes an object in the sense of M_i, that is, a function defined on the set A_i, and composes it with f_i, which yields a function on A_{i+1}, that is, an object in the sense of M_{i+1}. For instance, if

$$A(x_i) \text{ set } (x_i \in A_i)$$

is a set in the sense of M_i, then

$$f_i^*(A)(x_{i+1}) = A(f_i(x_{i+1})) \ (x_{i+1} \in A_{i+1}),$$

which is a set in the sense of M_{i+1}, and, if

$$a(x_i) \in A(x_i) \ (x_i \in A_i)$$

is an element of that set in the sense of M_i, then

$$f_i^*(a)(x_{i+1}) = a(f_i(x_{i+1})) \in A(f_i(x_{i+1})) \ (x_{i+1} \in A_{i+1}),$$

which is an element of the set $f_i^*(A)$ in the sense of M_{i+1}. The action of f_i^* on other types of objects is similar. Viewed proof theoretically, f_i^* is the translation from the theory T_i to the theory T_{i+1} given by the equation

$$\alpha_i = f_i(\alpha_{i+1}) \in A_i.$$

Thus f_i^* translates every symbol of T_i into the same symbol of T_{i+1}, except α_i which is translated into $f_i(\alpha_{i+1})$. Now, since substitution, and thereby composition, has the characteristic property of commuting with every operation, like Π, λ, app, etc., f_i^* is a homomorphism from M_i to M_{i+1}. This is what we should expect, since, what appears proof theoretically as a translation between theories, corresponds model theoretically to a homomorphism between models. Summing up, we have indeed to do with an inductive system of algebraic structures, in our case, models of type theory, which is a very rich kind of algebraic structure as compared with groups, rings, modules, or the like. Let

$$M_\alpha = \varinjlim M_i = \varinjlim M/A_i$$

be its inductive, or direct, limit.

According to the definition of an inductive limit, a set in the sense of M_α is a set in the sense of M_i, that is, a family of sets

$$A(x_i) \ (x_i \in A_i),$$

for some i, and it is defined to be equal to another such set, say

$$B(x_j) \ (x_j \in A_j),$$

whose index j may differ from i, provided

$$A(f_{ik}(x_k)) = B(f_{jk}(x_k)) \ (x_k \in A_k)$$

for some $k \geqslant \max(i,j)$, where I have put

$$f_{ik}(x_k) = f_i(f_{i+1}(\ldots f_{k-1}(x_k)\ldots)) \in A_i \ \ (x_k \in A_k)$$

for the sake of brevity. Likewise, an element of the set $A(x_i)$ $(x_i \in A_i)$ in the sense of M_α is a function

$$a(x_j) \in A(f_{ij}(x_j)) \ (x_j \in A_j)$$

for some $j \geqslant i$, and it is defined to be equal to another such element, say

$$b(x_k) \in A(f_{ik}(x_k)) \ (x_k \in A_k),$$

whose index $k \geqslant i$ may differ from j, provided

$$a(f_{jl}(x_l)) = b(f_{kl}(x_l)) \in A(f_{il}(x_l)) \ (x_l \in A_l)$$

at some stage $l \geqslant \max(j,k) \geqslant i$. Other types of objects in the sense of M_α are defined similarly. To take yet another example, which we shall need in the following, a family of sets over the set $A(x_i)$ $(x_i \in A_i)$ in the sense of M_α is a family of sets of two arguments

$$B(x_j,x) \ (x_j \in A_j, \ x \in A(f_{ij}(x_j)))$$

for some $j \geqslant i$, and it is defined to be equal to another such family, say

$$C(x_k,x) \ (x_k \in A_k, \ x \in A(f_{ik}(x_k))),$$

whose index $k \geqslant i$ may differ from j, provided

$$B(f_{jl}(x_l),x) = C(f_{kl}(x_l),x) \ (x_l \in A_l, \ x \in A(f_{il}(x_l)))$$

for some $l \geqslant \max(j,k) \geqslant i$.

So far, I have only explained how the various types are interpreted in M_α. I proceed to verify that M_α is a model of T_α. First of all, M_α becomes a nonstandard model of the standard theory T by letting every operation of T act pointwise, like in M_i, with the only extra complication that, since the operands begin to exist at different stages, in general, they have to be shifted out to a common later stage before the operation can be applied. For example, if

$$a(x_i) \in N \ (x_i \in A_i)$$

and

$$b(x_j) \in N \ (x_j \in A_j)$$

are two nonstandard natural numbers, that is, natural numbers in the sense of M_α, then their sum is

$$(a + b)(x_k) = a(f_{ik}(x_k)) + b(f_{jk}(x_k)) \in N \ (x_k \in A_k)$$

with $k = \max(i,j)$. The action of the other operations of the standard theory T on the objects of M_α is similar. It only remains to verify that the axioms that are proper to T_α, namely, the axioms

$$\begin{cases} \alpha_i \in A_i, \\ \alpha_i = f_i(\alpha_{i+1}) \in A_i \end{cases}$$

that govern the choice sequence α, become satisfied in M_α. The interpretation of the constant α_i is the identity function

$$x_i \in A_i \quad (x_i \in A_i).$$

Hence, in order to satisfy the definitional equality

$$\alpha_i = f_i(\alpha_{i+1}) \in A_i,$$

we must see to it that

$$f_{ij}(x_j) = f_i(f_{i+1\,j}(x_j)) \in A_i \quad (x_j \in A_j)$$

for some $j \geqslant \max(i,i+1) = i+1$. Clearly, it suffices to take $j = i+1$. Thus M_α is indeed a model of T_α. And it is not an arbitrarily contrived model: it is, to be sure, a nonstandard model of the standard theory, but it is the standard model, or intended interpretation, of the nonstandard theory.

Every standard object gives rise to a nonstandard object, namely, the function on A_0 which is constantly equal to the standard object in question. This is the analogue of the star embedding of classical nonstandard analysis.[13] For example,

[13] A. Robinson, op. cit., pp. 36-37.

a standard set A gives rise to the nonstandard set

$$A(x_0) = A \ (x_0 \in A_0),$$

a standard element a of A to the nonstandard element of $A(x_0)$ $(x_0 \in A_0)$ defined by the equation

$$a(x_0) = a \in A = A(x_0) \ (x_0 \in A_0),$$

and so on for other types of objects. This is an embedding of the standard model M into the nonstandard model M_α. To prove that it is injective, as an embedding should be, let A and B be two standard sets, and assume that their embeddings

$$A(x_0) = A \ (x_0 \in A_0)$$

and

$$B(x_0) = B \ (x_0 \in A_0)$$

are equal in the sense of M_α, that is, that

$$A(f_{0i}(x_i)) = B(f_{0i}(x_i)) \ (x_i \in A_i)$$

at some stage $i \geq 0$. Then, by invoking the element a_i which shows the set A_i to be nonempty, we can conclude, by substitution, that

$$A(f_{0i}(a_i)) = B(f_{0i}(a_i)).$$

On the other hand, again by substitution,

$$A(f_{0i}(a_i)) = A$$

and

$$B(f_{0i}(a_i)) = B,$$

so that, by symmetry and transitivity,

$$A = B.$$

The proof of the injectiveness of the star embedding of other types of objects is similar. Observe also the similarity with the proof of the model theoretic transfer principle given below. The star embedding was the last arrow to be explained in the commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\text{subtheory embedding}"] \arrow[d, "\text{intended interpretation}"'] & T_\alpha \arrow[d, "\text{intended interpretation}"] \\
M \arrow[r, "\text{star embedding}"'] & M_\alpha = \varinjlim M/A_i,
\end{tikzcd}$$

which summarizes the structure that we have erected.

Because of the identification of propositions and sets, a proposition in the sense of M_α, or a nonstandard proposition, is a propositional function on some A_i,

$$A(x_i) \text{ prop } (x_i \in A_i).$$

By definition, such a nonstandard proposition is nonstandardly true if it has a nonstandard proof, that is, if there exists an

$$a(x_j) \in A(f_{ij}(x_j)) \ (x_j \in A_j)$$

at some stage $j \geqslant i$. Because of the definition of the standard notion of truth, according to which truth is tantamount to the existence of a proof, this is clearly equivalent to requiring

$$A(f_{ij}(x_j)) \text{ true } (x_j \in A_j),$$

or, if you prefer,

$$(\forall x_j \in A_j) A(f_{ij}(x_j)) \text{ true},$$

at some stage $j \geq i$. Once the notion of nonstandard truth has been duly introduced, it is easy enough to establish the following model theoretic version of the transfer principle.

Transfer principle (model theoretic). Let A be a standard proposition. Then the embedding of A into the nonstandard model is nonstandardly true if and only if A is true in the standard sense.

To prove the sufficiency of the condition, let

$$A(x_0) = A \; (x_0 \in A_0)$$

be the embedding of the standard proposition A into the nonstandard model, and suppose

$$A \text{ true}$$

in the standard sense. Then, by weakening,

$$A \text{ true } (x_0 \in A_0),$$

and, by the principle that truth is preserved under definitional equality,

$$A(x_0) \text{ true } (x_0 \in A_0).$$

But

$$f_{00}(x_0) = x_0 \in A_0 \ (x_0 \in A_0),$$

so that, again by the same principle,

$$A(f_{0i}(x_i)) \text{ true } (x_i \in A_i)$$

already at stage $i = 0$, which shows that $A(x_0)$ $(x_0 \in A_0)$ is nonstandardly true.

Conversely, assume that $A(x_0)$ $(x_0 \in A_0)$ is nonstandardly true, that is, that

$$A(f_{0i}(x_i)) \text{ true } (x_i \in A_i)$$

at some stage $i \geq 0$. Then, since we have $a_i \in A_i$ at every stage i, we get

$$A(f_{0i}(a_i)) \text{ true}$$

by substitution. On the other hand, by substituting $f_{0i}(a_i)$ for x_0 in the definition of $A(x_0)$, we get

$$A(f_{0i}(a_i)) = A.$$

Hence, by preservation of truth under definitional equality, we can conclude

$$A \text{ true}$$

as desired.

If you compare this proof with the earlier proof of the proof theoretic version of the transfer principle, you will see that it is its exact model theoretic counterpart.

Commutation of nonstandard truth and the logical operations

Classical nonstandard analysis is built on the highly nonconstructive existence of an ultrafilter extending the Fréchet filter of all cofinal subsets of the set of natural numbers. One may wonder how we have been able to circumvent this in the preceding construction of the inductive limit model of nonstandard type theory, which roughly amounts to working with the Fréchet filter itself instead of an ultrafilter extending it. The answer seems to be that, however nonconstructive and nonstandard is classical nonstandard analysis, its interpretation of the logical operations is nevertheless standard, whereas, in nonstandard type theory, the logical operations receive a nonstandard interpretation, as they normally do in intuitionistic model theory, for instance, in Kripke semantics. And it is easier to construct a nonstandard model if it is allowed to be nonstandard throughout than if it is to be nonstandard in its interpretation of the notion of natural number and at the same time standard in its interpretation of the logical operations. Now, that a nonstandard model is standard in respect of the logical operations is tantamount to saying that nonstandard truth commutes with the logical operations. It is thus desirable to investigate to what extent, in our inductive limit interpretation, nonstandard truth commutes with the logical operations, which is to say, loosely speaking, exactly how nonstandard is its interpretation of the propositional connectives and the quantifiers. The result is the following.

<u>Theorem</u>. Of all the logical operations, nonstandard truth commutes with $\bot$, &, and $\exists$, but not with $\vee$, $\supset$, and $\forall$, in general.

By saying that nonstandard truth commutes with $\bot$, I mean of course that it is absurd that $\bot$ is nonstandardly true, that is, that $\bot$ is nonstandardly false. But $\bot$ is a standard proposition. Hence it follows directly from the model theoretic transfer principle that $\bot$ is nonstandardly true if and only if $\bot$ is true, which is manifestly not the case. Thus $\bot$ is indeed nonstandardly false.

The simplest way to prove that nonstandard truth commutes with conjunction is to note that, since the rule of conjunction introduction,

$$\frac{A \text{ true} \qquad B \text{ true}}{A \,\&\, B \text{ true}},$$

and the two rules of conjunction elimination,

$$\frac{A \,\&\, B \text{ true}}{A \text{ true}}, \qquad \frac{A \,\&\, B \text{ true}}{B \text{ true}},$$

are formally derivable in standard type theory, and the inductive limit interpretation is a nonstandard model of it, they must be validated by that interpretation. Hence A & B is nonstandardly true if and only if A is nonstandardly true and B is nonstandardly true. But it can also be checked directly as follows. Let

$$A(x_i) \ (x_i \in A_i)$$

and

$$B(x_j)\ (x_j \in A_j)$$

be two nonstandard propositions. Then their nonstandard conjunction is

$$A(f_{ik}(x_k))\ \&\ B(f_{jk}(x_k))\ (x_k \in A_k),$$

where $k = \max(i,j)$. Suppose that the two nonstandard conjuncts are nonstandardly true, that is, by the definition of nonstandard truth, that

$$A(f_{il}(x_l))\ \text{true}\ (x_l \in A_l)$$

and

$$B(f_{jm}(x_m))\ \text{true}\ (x_m \in A_m)$$

at some stages $l \geqslant i$ and $m \geqslant j$, respectively. Let $n = \max(l,m)$ be a common later stage. Then, by substitution,

$$A(f_{il}(f_{ln}(x_n)))\ \text{true}\ (x_n \in A_n)$$

and

$$B(f_{jm}(f_{mn}(x_n)))\ \text{true}\ (x_n \in A_n).$$

On the other hand,

$$f_{il}(f_{ln}(x_n)) = f_{in}(x_n) = f_{ik}(f_{kn}(x_n)) \in A_i\ (x_n \in A_n)$$

and

$$f_{jm}(f_{mn}(x_n)) = f_{jn}(x_n) = f_{jk}(f_{kn}(x_n)) \in A_j\ (x_n \in A_n).$$

Hence, by preservation of truth under definitional equality and conjunction introduction,

$$A(f_{ik}(f_{kn}(x_n))) \mathbin{\&} B(f_{jk}(f_{kn}(x_n))) \text{ true } (x_n \in A_n)$$

for $n = \max(l,m) \geqslant \max(i,j) = k$, which is to say that the nonstandard conjunction of the two nonstandard propositions is nonstandardly true. Conversely, assume that this is the case, that is, by the definition of nonstandard truth, that

$$A(f_{ik}(f_{kl}(x_l))) \mathbin{\&} B(f_{jk}(f_{kl}(x_l))) \text{ true } (x_l \in A_l)$$

at some stage $l \geqslant k = \max(i,j)$. Then, by conjunction elimination and preservation of truth under definitional equality,

$$A(f_{il}(x_l)) \text{ true } (x_l \in A_l)$$

with $l \geqslant i$, and

$$B(f_{jl}(x_l)) \text{ true } (x_l \in A_l)$$

with $l \geqslant j$, which shows that the two nonstandard conjuncts are both nonstandardly true.

To complete the proof of the positive part of the theorem, we must convince ourselves that nonstandard truth also commutes with existence. The simple way to do it is to note that the usual rule of existence introduction,

$$\frac{a \in A \qquad B(a) \text{ true}}{(\exists x \in A)B(x) \text{ true}},$$

as well as the strong rules of existence elimination,

$$\frac{c \in (\exists x \in A)B(x)}{p(c) \in A}, \qquad \frac{c \in (\exists x \in A)B(x)}{B(p(c)) \text{ true}},$$

are all derivable in standard type theory, so that they must be validated by the nonstandard model. Hence, if B(a) is nonstandardly true for a nonstandard element a of the nonstandard set A, then the nonstandard existential proposition $(\exists x \in A)B(x)$ is nonstandardly true. Conversely, assume that $(\exists x \in A)B(x)$ is nonstandardly true. By the definition of nonstandard truth, this means that it has a nonstandard proof c. Then the left projection p(c) of that proof in the sense of the nonstandard model is a nonstandard element of the nonstandard set A such that the nonstandard proposition B(p(c)) is nonstandardly true. Thus nonstandard truth commutes with existence. The more laborious direct verification proceeds as follows. Let

$$A(x_i) \ (x_i \in A_i)$$

be a nonstandard set, and let

$$B(x_j,x) \ (x_j \in A_j,\ x \in A(f_{ij}(x_j)))$$

with $j \geq i$ be a nonstandard propositional function over it. Quantifying it existentially in the sense of the nonstandard model, we get the nonstandard proposition

$$(\exists x \in A(f_{ij}(x_j)))B(x_j,x) \ (x_j \in A_j).$$

Let

$$a(x_k) \in A(f_{ik}(x_k)) \ (x_k \in A_k)$$

with $k \geq i$ be a nonstandard element of the nonstandard set A. Then B(a) in the sense of the nonstandard model is the nonstandard proposition

$$B(f_{jl}(x_l),a(f_{kl}(x_l))) \ (x_l \in A_l),$$

where $l = \max(j,k)$. Assume it to be true, that is, assume

$$B(f_{jm}(x_m),a(f_{km}(x_m))) \text{ true } (x_m \in A_m)$$

at some stage $m \geq l$. Then, by standard existence introduction,

$$(\exists x \in A(f_{im}(x_m)))B(f_{jm}(x_m),x) \text{ true } (x_m \in A_m)$$

for $m \geq l = \max(j,k) \geq j$, which shows that the nonstandard existential proposition is nonstandardly true. Conversely, assume that the nonstandard existential proposition is nonstandardly true, that is, according to the definition of the notion of nonstandard truth, that it has a nonstandard proof

$$c(x_k) \in (\exists x \in A(f_{ik}(x_k)))B(f_{jk}(x_k),x) \ (x_k \in A_k)$$

with $k \geq j \geq i$. By the strong rules of existence elimination, we can conclude from this that

$$p(c(x_k)) \in A(f_{ik}(x_k)) \ (x_k \in A_k)$$

and

$$B(f_{jk}(x_k),p(c(x_k))) \text{ true } (x_k \in A_k).$$

Thus we have found a nonstandard element of the nonstandard set $A(x_i)$ $(x_i \in A_i)$ which satisfies the nonstandard propositional function $B(x_j,x)$ $(x_j \in A_j,\ x \in A(f_{ij}(x_j)))$. This finishes the proof of the positive part of the theorem.

To prove the negative part of the theorem, I shall make use of the projective system

$$A_0 \xleftarrow[\text{id}]{} A_0 \xleftarrow[\text{id}]{} \cdots \xleftarrow[\text{id}]{} A_0 \xleftarrow[\text{id}]{} A_0 \xleftarrow[\text{id}]{} \cdots,$$

where A_0 is a fixed nonempty set, that is, a set containing an element $a_0 \in A_0$. Let

$$\begin{aligned} ? = ?_0 = \text{id}(?_1), & \\ ?_1 = \text{id}(?_2), & \\ ?_2 = \text{id}(?_3), & \\ \ldots & \end{aligned}$$

be the choice sequence that it defines. Furthermore, let $A(x_0)$ and $B(x_0)$ be two standard propositional functions of the variable $x_0 \in A_0$. Then the nonstandard proposition $A(?)$ is nonstandardly true provided $A(x_0)$ is true for $x_0 \in A_0$, or, equivalently, $(\forall x_0 \in A_0)A(x_0)$ is true, in the standard sense. Likewise, $B(?)$ is nonstandardly true provided $(\forall x_0 \in A_0)B(x_0)$ is true in the standard sense. Now, consider the nonstandard disjunctive proposition $A(?) \vee B(?)$. It is nonstandardly true provided $(\forall x_0 \in A_0)(A(x_0) \vee B(x_0))$ is true in the standard sense. Hence, since the implication in the judgement

$$\begin{aligned} &(\forall x_0 \in A_0)A(x_0) \vee (\forall x_0 \in A_0)B(x_0) \\ &\supset (\forall x_0 \in A_0)(A(x_0) \vee B(x_0)) \text{ true} \end{aligned}$$

cannot be reversed, in general, nonstandard truth does not commute with disjunction. Another example of the failure of nonstandard truth to commute with disjunction will be given later.

Next, consider the nonstandard implicative proposition $A(?) \supset B(?)$. It is nonstandardly true provided $(\forall x_0 \in A_0)(A(x_0) \supset B(x_0))$ is true in the standard sense. But the outermost implication in the judgement

$$(\forall x_0 \in A_0)(A(x_0) \supset B(x_0))$$
$$\supset ((\forall x_0 \in A_0)A(x_0) \supset (\forall x_0 \in A_0)B(x_0)) \text{ true}$$

cannot be reversed, in general. Hence, if $A(?) \supset B(?)$ is nonstandardly true, the nonstandard truth of $A(?)$ entails the nonstandard truth of $B(?)$, but not conversely, which shows that nonstandard truth fails to commute with implication.

Finally, let $A(x_0)$ be a set depending on the variable $x_0 \in A_0$, that is, a family of sets over A_0, and $B(x_0,x)$ a propositional function of the two variables $x_0 \in A_0$ and $x \in A(x_0)$, both in the standard sense, and consider the nonstandard proposition $(\forall x \in A(?))B(?,x)$. By the definition of nonstandard truth, it is nonstandardly true provided $(\forall x_0 \in A_0)(\forall x \in A(x_0))B(x_0,x)$ is true in the standard sense. On the other hand, an arbitrary nonstandard element of $A(?)$ is of the form $a(?)$, where $a(x_0) \in A(x_0)$ for $x_0 \in A_0$ is a standard function, and $B(?,a(?))$ is nonstandardly true provided $(\forall x_0 \in A_0)B(x_0,a(x_0))$ is true in the standard sense. Hence the nonstandard truth of $(\forall x \in A(?))B(?,x)$ entails the nonstandard truth of $B(?,a(?))$ for all nonstandard elements $a(?)$ of $A(?)$, but not conversely, since the implication in the judgement

$$(\forall x_0 \in A_0)(\forall x \in A(x_0))B(x_0,x)$$
$$\supset (\forall z \in (\Pi x_0 \in A_0)A(x_0))(\forall x_0 \in A_0)B(x_0,\mathrm{app}(z,x_0)) \text{ true}$$

cannot be reversed, in general. Thus nonstandard truth fails to commute with universal quantification, and the proof of the theorem is finished.

Miscellaneous examples

This is not the place to begin a systematic development of intuitionistic nonstandard analysis, but a few examples may help to give an idea of what can be done with the nonstandard concepts.

Example 1. Make the pair of definitions

$$\begin{cases} \mathrm{Even}(a) = (\exists y \in N)I(N, a, 2\cdot y), \\ \mathrm{Odd}(a) = (\exists y \in N)I(N, a, 2\cdot y + 1), \end{cases}$$

where $a \in N$. Then the judgement

$$\mathrm{Even}(x) \vee \mathrm{Odd}(x) \text{ true } (x \in N)$$

is easily proved by induction on x in standard type theory. Hence, substituting ∞ for x, we can derive

$$\mathrm{Even}(\infty) \vee \mathrm{Odd}(\infty) \text{ true}$$

in the nonstandard extension. Semantically, this means that the nonstandard proposition $\mathrm{Even}(\infty) \vee \mathrm{Odd}(\infty)$ is nonstandardly true. On the other hand, at no stage i = 0, 1, etc., do we have

$$\mathrm{Even}(s^i(x)) \text{ true } (x \in N),$$

nor do we have

$$\mathrm{Odd}(s^i(x)) \text{ true } (x \in N),$$

which is to say that the two nonstandard propositions $\mathrm{Even}(\infty)$ and $\mathrm{Odd}(\infty)$ are both nonstandardly false. We have here another, more memorable example of the failure of nonstandard truth to commute with disjunction. In classical nonstandard analysis, one of the two nonstandard propositions Even(∞) and Odd(∞) is forced to be true, which one of them depends on the choice of the ultrafilter extending the Fréchet filter of cofinal subsets of the set of natural numbers. Since this choice is anyway left arbitrary, it seems more natural not to force any one of Even(∞) and Odd(∞) to be true. This is made possible by the nonstandard interpretation of disjunction in the present, intuitionistic version of nonstandard analysis.

Example 2. Let $A(x)$ ($x \in \mathbb{N}$) be a standard property of natural numbers. Then $A(\infty)$ is a nonstandard proposition, whose nonstandard truth entails the nonstandard truth of $A(a)$ for all infinite $a \in \mathbb{N}$.

To see this, assume

$$A(\infty) \text{ true.}$$

By the definition of nonstandard truth, this means that

$$A(s^i(x)) \text{ true } (x \in \mathbb{N})$$

at some stage $i = 0, 1$, etc. If we define addition of natural numbers by recursion on the first argument,

$$\begin{cases} 0 + b = b \in \mathbb{N}, \\ s(a) + b = s(a + b) \in \mathbb{N}, \end{cases}$$

we have

$$s^i(x) = s^i(0) + x \in \mathbb{N} \ (x \in \mathbb{N}),$$

so that we can pass to

$$A(s^i(0) + x) \text{ true } (x \in \mathbb{N})$$

by preservation of truth under definitional equality. From this, we get

$$I(N, s^i(0) + x, y) \supset A(y) \text{ true } (y \in N)$$

by I-elimination and $\supset$-introduction. Standard rules of intuitionistic predicate logic now yield

$$(\forall y \in N)((\exists x \in N)I(N, s^i(0) + x, y) \supset A(y)) \text{ true.}$$

But

$$\begin{aligned} &(\forall y \in N)((\exists x \in N)I(N, s^i(0) + x, y) \supset A(y)) \\ &= (\forall y \in N)(s^i(0) \leqslant y \supset A(y)) \\ &= (\forall y \geqslant s^i(0))A(y) \end{aligned}$$

by definition, so that, again by preservation of truth under definitional equality,

$$(\forall y \geqslant s^i(0))A(y) \text{ true.}$$

Now, let $a \in N$ be an arbitrary infinite in the sense of nonstandard arithmetic, which is to say that

$$s^j(0) \leqslant a \text{ true}$$

for all $j = 0, 1$, etc., so that, in particular,

$$s^i(0) \leqslant a \text{ true.}$$

Since the standard logical laws continue to hold in the nonstandard interpretation, we can then conclude

$$A(a) \text{ true}$$

by $\forall$- and $\supset$-elimination. Thus the nonstandard truth of $A(\infty)$ entails that of $A(a)$ for an arbitrary infinite $a \in N$.

<u>Example 3</u>. We already know that each ∞_i is infinite, but what does an arbitrary infinite $a \in N$ look like? By the definition of the nonstandard model,

$$a = f(\infty_i) \in N$$

for some $i = 0, 1$, etc., where

$$f(x) \in N \ (x \in N)$$

is a standard number theoretic function. On the other hand, that $a = f(\infty_i) \in N$ is infinite in the sense of nonstandard arithmetic means, by definition, that

$$s^j(0) \leqslant f(\infty_i) \text{ true}$$

for all $j = 0, 1$, etc. By the definition of nonstandard truth, this means, in turn, that, for all $j = 0, 1$, etc., there exists a stage $k = 0, 1$, etc., such that

$$s^j(0) \leqslant f(s^k(x)) \text{ true } (x \in N).$$

Thus an infinite natural number is the image of an ∞_i under a standard number theoretic function which grows beyond all bounds.

<u>Example 4</u>. Define a propositional function $P(x)$ of the variable $x \in N$ by the recursion equations

$$\begin{cases} P(0) = \perp, \\ P(s(x)) = \neg P(x). \end{cases}$$

This is easily done in standard type theory by means of the universe axioms. In fact, the definition

$$P(x) = T(\mathrm{rec}(x, \perp, (x,y)\sim y))$$

will do. Since we have $\infty \in N$ in the nonstandard extension, $P(\infty)$ is a nonstandard proposition, and

$$P(\infty) = P(s(\infty_1)) = \sim P(\infty_1) = \sim P(s(\infty_2)) = \sim\sim P(\infty_2) = \ldots$$

Thus, when it is evaluated lazily, $P(\infty)$ appears as an endless sequence of negation signs. Is $P(\infty)$ nonstandardly true or nonstandardly false? Suppose it to be nonstandardly true. By definition, this means that

$$P(s^i(x)) \text{ true } (x \in N)$$

at some stage $i = 0, 1$, etc. This entails that both $P(s^i(0))$ and $P(s^i(s(0)))$ are true at that stage. But

$$P(s^i(s(0))) = P(s(s^i(0))) = \sim P(s^i(0)),$$

so that both $P(s^i(0))$ and $\sim P(s^i(0))$ would be true, which is impossible. Thus $P(\infty)$ is nonstandardly false. This being so, $P(\infty_i)$ is actually nonstandardly false for all $i = 0, 1$, etc., although $P(\infty_i) = P(s(\infty_{i+1})) = \sim P(\infty_{i+1})$. There is no contradiction in this: it only shows that nonstandard truth fails to commute with negation, which comes as no surprise. Indeed, negation is defined by the equation $\sim A = A \supset \perp$, and we already know that nonstandard truth fails to commute with implication.

Example 5. I shall construct a nonstandard element of List(N) which produces the stream of integers

$$(0,(s(0),(s(s(0)),\ldots)))$$

when it is evaluated lazily. To this end, define the standard function

$$f(l,n) \in \mathrm{List}(\mathrm{N}) \quad (l \in \mathrm{N},\ n \in \mathrm{N})$$

by the pair of equations

$$\begin{cases} f(0,n) = \mathrm{nil} \in \mathrm{List}(\mathrm{N}), \\ f(s(l),n) = (n,f(l,s(n))) \in \mathrm{List}(\mathrm{N}). \end{cases}$$

This is a double recursion, but, putting

$$f(l,n) = \mathrm{app}(g(l),n),$$

it reduces to the primitive recursion

$$\begin{cases} g(0) = (\lambda n)\mathrm{nil} \in \mathrm{N} \to \mathrm{List}(\mathrm{N}), \\ g(s(l)) = (\lambda n)(n,\mathrm{app}(g(l),s(n))) \in \mathrm{N} \to \mathrm{List}(\mathrm{N}), \end{cases}$$

which has the solution

$$g(l) = \mathrm{rec}(l,(\lambda n)\mathrm{nil},(x,y)(\lambda n)(n,\mathrm{app}(y,s(n)))) \\ \in \mathrm{N} \to \mathrm{List}(\mathrm{N})$$

in standard type theory. We can now use the axiom $\infty \in \mathrm{N}$ to derive

$$f(\infty,0) \in \mathrm{List}(\mathrm{N})$$

in the nonstandard extension. Evaluating $f(\infty,0)$ lazily, we get

$$\begin{aligned} f(\infty,0) &= f(s(\infty_1),0) \\ &= (0,f(\infty_1,s(0))) = (0,f(s(\infty_2),s(0))) \\ &= (0,(s(0),f(\infty_2,s(s(0))))) = \ldots \end{aligned}$$

Thus the stream of integers is produced.[14]

Example 6. Let a projective system of nonempty sets of the sort that underlies the inductive limit interpretation be given internally, that is, suppose

$$\begin{cases} A(n) \text{ set } (n \in \mathbb{N}), \\ f(n,x) \in A(n) \ (n \in \mathbb{N},\ x \in A(s(n))), \\ a(n) \in A(n) \ (n \in \mathbb{N}) \end{cases}$$

in the sense of the nonstandard model. Then we can construct an externally indexed sequence of elements

$$\alpha_i \in A(s^i(0))$$

which satisfy the equations

$$\alpha_i = f(s^i(0), \alpha_{i+1}) \in A(s^i(0))$$

for i = 0, 1, etc. (We cannot, in general, have such a sequence internally without running into contradiction.)

To see this, define first an auxiliary function

$$g(l,n) \in A(n) \ (l \in \mathbb{N},\ n \in \mathbb{N})$$

by the equations

$$\begin{cases} g(0,n) = a(n) \in A(n), \\ g(s(l),n) = f(n, g(l,s(n))) \in A(n). \end{cases}$$

[14] The idea of dealing with stream computation by introducing an infinite number is independently due to S. Goto, Nonstandard normalization, US-Japan Workshop, Honolulu, May 1987. He proposes to apply Robinson's nonstandard analysis to interpret number theory extended by a new constant for the infinite number.

This is a double recursion, but it is readily solved in standard type theory by putting

$$g(l,n) = \mathrm{app}(h(l),n) \in A(n),$$

where

$$h(l) \in (\Pi n \in N)A(n) \quad (l \in N)$$

is defined by the primitive recursion

$$\begin{cases} h(0) = (\lambda n)a(n) \in (\Pi n \in N)A(n), \\ h(s(l)) = (\lambda n)f(n,\mathrm{app}(h(l),s(n))) \in (\Pi n \in N)A(n), \end{cases}$$

which has the solution

$$h(l) = \mathrm{rec}(l,(\lambda n)a(n),(x,y)(\lambda n)f(n,\mathrm{app}(y,s(n)))) \\ \in (\Pi n \in N)A(n).$$

We can now put

$$\alpha_i = g(\infty_i, s^i(0)) \in A(s^i(0))$$

for i = 0, 1, etc. This is an external sequence which satisfies

$$\begin{aligned} \alpha_i &= g(\infty_i, s^i(0)) \\ &= g(s(\infty_{i+1}), s^i(0)) \\ &= f(s^i(0), g(\infty_{i+1}, s(s^i(0)))) \\ &= f(s^i(0), g(\infty_{i+1}, s^{i+1}(0))) \\ &= f(s^i(0), \alpha_{i+1}) \in A(s^i(0)) \end{aligned}$$

as desired. This example explains the canonical character of the choice sequence $\infty = s(\infty_1) = s(s(\infty_2))$ = etc., because, once we have access to it, we can define any other choice sequence associated with a projective system of nonempty sets, provided only that it is given internally.

Example 7. Define by recursion a family of sets $B(n)$ $(n \in N)$ satisfying the equations

$$\begin{cases} B(0) = N_1, \\ B(s(n)) = B(n) + B(n). \end{cases}$$

The universe axioms allow you to do this in standard type theory. Indeed, it suffices to put

$$B(n) = T(rec(n, n_1, (x,y)(y + y))).$$

Consider now the nonstandard set $B(\infty)$. It satisfies the equations

$$\begin{aligned} B(\infty) &= B(s(\infty_1)) \\ &= B(\infty_1) + B(\infty_1) = B(s(\infty_2)) + B(s(\infty_2)) \\ &= (B(\infty_2) + B(\infty_2)) + (B(\infty_2) + B(\infty_2)) \\ &= \ldots \end{aligned}$$

Thus $B(\infty)$ can be endlessly divided into two equal halves: it is a nonstandard version of the Cantor space.[15] Let Q denote the standard set of rational numbers. I shall show how to define the integral of a function

$$f \in B(\infty) \to Q,$$

which is as nonstandard as the set to which it belongs, of course, with respect to the usual uniform distribution. To this

[15] For a nonstandard version of the Cantor space in classical nonstandard analysis, see S. Albeverio et al., op. cit., p. 65.

end, define the sequence of sums

$$S(n,f) \in Q \ (n \in N,\ f \in B(n) \rightarrow Q)$$

by the recursive equations

$$\begin{cases} S(0,f) = \mathrm{app}(f,0_1), \\ S(s(n),f) = S(n,(\lambda x)\mathrm{app}(f,i(x))) + S(n,(\lambda y)\mathrm{app}(f,j(y))). \end{cases}$$

These are easily solved in standard type theory by putting

$$S(n,f) = \mathrm{app}(F(n),f),$$

where

$$F(n) \in (B(n) \rightarrow Q) \rightarrow Q \ (n \in N)$$

is defined by the primitive recursion

$$\begin{cases} F(0) = (\lambda f)\mathrm{app}(f,0_1), \\ F(s(n)) = (\lambda f)(\mathrm{app}(F(n),(\lambda x)\mathrm{app}(f,i(x))) \\ \qquad\qquad + \mathrm{app}(F(n),(\lambda y)\mathrm{app}(f,j(y)))). \end{cases}$$

The analogue of the sequence of Riemann sums

$$I(n,f) \in Q \ (n \in N,\ f \in B(n) \rightarrow Q)$$

is defined by putting

$$I(n,f) = \frac{S(n,f)}{2^n},$$

that is, by dividing $S(n,f)$ by the total number of elements of the set $B(n)$. We can now express the searched for integral of $f \in B(\infty) \rightarrow Q$ simply as

$$I(\infty, f) = \frac{S(\infty, f)}{2^{\infty}}.$$

This is a nonstandard rational number. Let us see what the sequence of rational approximations is of which it is the limit. By the definition of the nonstandard model, the meaning of $f \in B(\infty) \to Q$ is that

$$f = g(\infty_i) \in B(s^i(\infty_i)) \to Q = B(\infty) \to Q,$$

where

$$g(x) \in B(s^i(x)) \to Q \quad (x \in N)$$

is a standard function. Hence

$$I(\infty, f) = I(\infty, g(\infty_i)) = I(s^i(\infty_i), g(\infty_i)),$$

which is the value at ∞_i of the standard sequence of rational numbers

$$I(s^i(x), g(x)) \in Q \quad (x \in N).$$

This means that the rational approximations exist from stage i and onwards. At stage j = i, i+1, etc., the rational approximation is obtained by running the program

$$I(s^j(0), g(s^{j-i}(0))) \in Q.$$

Convergence is another matter.